Combinatorial Networks
Week 4, Wednesday

An application of 4-uniform hypergraph on geometry

- **Theorem.** (Erdös-Szekeres 1935) For any $m \geq 4$, there is a least $N := N(m)$ such that for any N points in plane (which are in general position), there are m points in convex position.
- **Theorem 1.** $N(m) \leq R^{(4)}(5, m)$.
- **Proof.** Let $n = R^{(4)}(5, m)$. For any given n points in plane, we want to find m points in convex position. Define a $K_n^{(4)}$ on the n points and define a 2-edge-coloring
$$c : \binom{[n]}{4} \to \{blue, red\}$$
by: $c(\{ijkl\})$ = blue, if i, j, k, l are in convex position, otherwise color red.
Since $n = R^{(4)}(5, m)$, this 2-edge-coloring c has either a red $K_5^{(4)}$ or a blue $K_m^{(4)}$. If there is a red $K_5^{(4)}$, then there are 5 points such that any 4 points are not in convex position, which contradicting to fact 1. So we must have $K_m^{(4)}$, which means there are m points such that any 4 points are in convex position, then those m points are in convex position. ∎
- **Theorem 2.** $N(m) \leq R^{(3)}(m, m)$.
- **Proof.** Given $n = R^{(3)}(m, m)$ points in plane, we label them as $1, 2, ..., n$. For $i < j < k$, define
$$c(\{ijk\}) = \begin{cases} \text{blue, if } i, j, k \text{ are clockwise,} \\ \text{red, if } i, j, k \text{ are counter clockwise.} \end{cases}$$
Note: Any non-convex polygon can NOT have all triangles clockwise (or counter clockwise). From this fact, we can obtain theorem 2. ∎
- **Fact.** $N(3) = 3, N(4) = 5, N(5) = 9, N(6) = 17$.
- **Conjecture.** (The happy ending problem) $N(m) = 2^{m-2} + 1$.
- **Best bounds.** $2^{m-2} + 1 \leq N(m) \leq \binom{2m-4}{m-2} \approx 4^m$.

The second proof on Hypergraph Ramsey Number: The stepping up argument

- **Theorem.** $R^{(r)}(s, t) \leq 2^{\binom{R^{(r-1)}(s-1,t-1)}{r-1}}$.
- **Proof.** In fact, we prove $r = 3$, i.e. $R^{(3)}(s, t) \leq 2^{\binom{R(s-1,t-1)}{2}}$.
Let $m = R(s-1, t-1) + 1$ and $N = 2^{\binom{m-1}{2}}$, consider $K_N^3 = (V, \binom{V}{3})$, consider its any 2-edge-coloring, we want to find a blue K_s^3 or a red K_t^3.
Definition: Vertices $v_1, v_2, ..., v_n$ are feasible for c, if there is $\chi : \binom{[n]}{2} \to$ {red,blue} such that for any i, j, k,
$$\chi(i, j) = c(\{i, j, k\})(*).$$

Definition: For a set $S_n \subset V - \{v_1, v_2, ..., v_n\}$, $v_1, v_2, ..., v_n, S_n$ are feasible for c, if for any $u \in S_n$, $v_1, v_2, ..., v_n, u$ are feasible for c.
Lemma: If there is a feasible $v_1, v_2, ..., v_m$ of length m, then there is a blue K_s^3 or a red K_t^3.
Proof of lemma: Consider $v_1, v_2, ..., v_{m-1}$, there is $\chi : \binom{[m-1]}{2} \to \{\text{red,blue}\}$ such that $(*)$ holds, then χ has a blue K_{s-1} or a red K_{t-1}. If there is a blue K_{s-1}, say vertices $v_{i_1}, v_{i_2}, ..., v_{i_{s-1}}$, then $v_{i_1}, v_{i_2}, ..., v_{i_{s-1}}, v_m$ form a blue $K_s^{(3)}$ by $(*)$. ∎
We will inductively construct feasible $v_1, v_2, ..., v_i, S_i$, where $S_i \subset V - \{v_1, v_2, ..., v_i\}$ for $1 \le i \le m-1$.
At round 1, pick any $v_1 \in V$ and denote $S_1 = V - \{v_1\}$, so v_1, S_1 are feasible. Assume that $v_1, v_2, ..., v_i, S_i$ are feasible, pick $v_{i+1} \in S_i$ arbitrarily. For any $u \in S_i - \{v_{i+1}\}$, define a vector $\vec{v}(u) = (c_1, c_2, ..., c_i)$ where $c_j = c(\{v_j, v_{i+1}, u\})$ for $1 \le j \le i$. For $u_1, u_2 \in S_i - \{v_{i+1}\}$, we say $u_1 \sim u_2$ are equivalent if $\vec{v}(u_1) = \vec{v}(u_2)$. Note that $\vec{v}(u)$ has 2^i choices, so we can partition $S_i - \{v_{i+1}\}$ into 2^i equivalent classes. Let S_{i+1} be the largest one of equivalent classes, namely, $|S_{i+1}| \ge 2^{-i}(|S_i| - 1)$, note that $v_1, v_2, ..., v_{i+1}, S_{i+1}$ are still feasible. Let $s_i = |S_i|$. If $s_{m-1} \ge 1$, then $v_1, v_2, ..., v_m$ are feasible.

$$\begin{aligned}
s_{m-1} &\ge 2^{-(m-2)} s_{m-2} - 2^{-(m-2)} \\
&\ge 2^{-(m-2)}(2^{-(m-3)} s_{m-3} - 2^{-(m-3)}) - 2^{-(m-2)} \\
&\ge ...... \\
&\ge 2^{-[(m-2)+(m-3)+...+1]} s_1 - \sum_{j=1}^{m-2} 2^{-\sum_{i=j}^{m-2}} \\
&= 2^{-\binom{m-1}{2}} s_1 - \sum_{j=1}^{m-2} 2^{-\sum_{i=j}^{m-2}} \\
&= 2^{-\binom{m-1}{2}} N - 2^{-\binom{m-1}{2}} - \sum_{j=1}^{m-2} 2^{-\sum_{i=j}^{m-2}} > 0
\end{aligned}$$

So $s_{m-1} > 0$, there is a feasible sequence $v_1, v_2, ..., v_m$. ∎